SIG 787 Mathematics for AI
Assignment No: 2
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(4)	find eigenvalues and eigenvectors of the matrix $G = AA^T$, and orthonormalise them. Call them u_1, u_2, and u_3 according to the order of $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of eigenvalues of G. Construct the matrix $U = [u_1 \; u_2 \; u_3]$, and check whether it is an orthogol matrix.	11
(5)	find three vectors $\{w_1, w_2, w_3\}$ so that $w_i = \frac{1}{\sigma_i} A v_i$, $i = 1, 2$; $w_3 \perp w_1$, $w_3 \perp w_2$, and $\lVert w_3 \rVert = 1$. Compare $\{w_1, w_2, w_3\}$ with $\{u_1, u_2, u_3\}$ and comment similar characteristics of these sets.	15

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(7)	Do the questions (1), (2), (3) (4) for the slightly different version of A $A_P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$ Construct $\sigma_{pi}, v_p, u_p, V_p, D_p$ and U_p. Justify the changes you observe. (In this question you may use decimal numbers.)	23
(8)	Construct a low-rank approximation of A and A_p following this definition using their corresponding eigenvectors. $A \approx \sigma_1 v_1 u_1^T$ $A_p \approx \sigma_{p1} v_{p1} u_{p1}^T$ here σ_{p1}, v_{p1} and u_{p1} are corresponding to the eigenvalues and eigenvectors related to A_p. compare these two matrix & justify change observed.	37
(9)	(for D and HD levels) Considered slightly altered version of A	41
(9.1)	Find eigenvalue & eigenvectors of A_ϵ, A_ϵ^T, and $A_\epsilon^T A_\epsilon$ in term of ϵ. $A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix}$ Compute $\sigma_i = \sqrt{\lambda_i}$ in terms of ϵ for positive eigen values.	
(9.2)	Construct V_ϵ, D_ϵ, and U_ϵ	
(9.3)	Critically analyse the impact of ϵ on σ_i, λ_i, eigenvectors, V_ϵ, D_ϵ, and U_ϵ.	

Question 1) Consider the following matrix. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ find rank (A), a basis for the column space of A, C(A), a basis for the row space of A, $C(A^T)$, a basis for the null space of A, N(A), and a basis for the left null space of A, $N(A^T)$

Hint: for a $m \times n$ matrix A, the column space of A is defined as $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$, and the row space of A is defined as $C(A^T) = \{A^T y \mid y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$. Also, the nullspace N(A) is defined as $\{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$, for the same matrix, the left nullspace $N(A^T)$ is defined as $\{y \in \mathbb{R}^m \mid A^T y = 0\} \subseteq \mathbb{R}^m$. Also we have

$$\dim(C(A)) + \dim(N(A^T)) = m$$
$$\dim(C(A^T)) + \dim(N(A)) = n.$$

[Note. You need to use gaussion elimination to answer this question. For all the linear systems in this assignment, you need to use Gaussian elimination to solve them.]

Solution: Given:-

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is already in row echelon form (upper trinagular form).

Gaussian Elimination:-

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Row 1 already has a leading 1
- In Row 2, the leading entry is also 1 and its in a new column
- Row 3 is all zeros → nothing to eliminate

1. find Rank(A):

- The rand of A is the number of linearly independent columns (for rows) in A.
- The colums of A are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which are linearly independent.
- The third row is all zeros, so it does not contribute to the rank.
- The rank is the number of leading 1s (pivots) in the row echlon form of A, matrix A is already in row echelon form.

Answer 1 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ The rank of rank(A) = Rank(A) = 2

2. Basis for the Collumn Space C(A):

- Column space is spanned by the linearly independent columns of A.

The columns of A are

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

These are linearly independent.

So

Answer: 2

$$\boxed{\text{Basis for } C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

3. Basis for the Row Space $C(A^T)$:

Row space of A is the span of the non-zero rows.

non-zero rows:

$$[1, 0], [0\ 1]$$

So

Answer 3:

$$\boxed{\text{Basis for } C(A^T) = \{[1, 0], [0\ 1]\}}$$

4. Basis for the null space $N(A)$:

The null space consists of all vectors x such that $Ax = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives $x_1 = 0$
$x_2 = 0$

Thus the only solution is the zero vector, meaning the null space is trivial.

So.

Answer: 4 : Basis of the null space $N(A)$
The only solution is the zero vector.

$N(A) = \{0\}$; Basis $= \emptyset$

5 Basis for the left null space $N(A^T)$:
The left null space consists of all vectors y such that $A^T y = 0$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives.

$y_1 = 0$
$y_2 = 0$

y_3 is free.
Thus the null space is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

Answer 5

Basis for left null space $N(A^T)$:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Answer & Solution for Question 1 is

- rank$(A) = 2$
- Basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
- Basis for $C(A^T)$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
- Basis for $C(A^T)$: $\left\{ [1\ 0], [0\ 1] \right\}$
- Basis for $N(A)$: $\emptyset$ (only zero vector)
- Basis for $N(A^T)$: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

The dimensions satisfy

- $\dim(C(A)) + \dim(N(A^T)) = 2 + 1 = 3 = m$
- $\dim(C(A^T)) + \dim(N(A)) = 2 + 0 = 2 = n$.

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Question no: 2 : Construct $B = A^T A$, and find
(2) eigenvalues and eigenvectors of B. for positive eigenvalues of B, define $\sigma_i = \sqrt{\lambda_i}$, where $\lambda_1 \geq \lambda_2$. Construct matrix D as follow.

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

Solution for Questionno: 2

given the matrix : $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Construct $B = A^T A$
first compute A^T

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

now, multiplying A^T by A:

$$B = A^T A = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} (1)(1) + (0)(0) + (0)(0) & (1)(0) + (0)(1) + (0)(0) \\ (0)(1) + (1)(0) + (0)(0) & (0)(0) + (1)(1) + (0)(0) \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Step 2: find Eigenvalues and Eigenvectors of B

The matrix B is already diagonal, so its eigenvalues are the diagonal entries.

$$\boxed{\lambda_1 = 1 \qquad \lambda_2 = 1}$$

The eigenvectors corresponding to $\lambda = 1$ are any non-zero vectors in $\mathbb{R}^2$, since B is the identity matrix A standard choice for the basis of the eigenspace is.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Step 3 Define $\sigma_i = \sqrt{\lambda_i}$

Since both eigenvalues are positive ($\lambda_1 = 1$; $\lambda_2 = 1$)

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

- Step 4 Construct Matrix D.

The matrix D is constructed as:

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So answer for Question 2

(2) Construct $B = A^T A$, and find eigenvalues and eigenvectors of B. for positive eigenvalues of B, define $\sigma_i = \sqrt{\lambda_i}$, where $\lambda_1 \geq \lambda_2$

Construct matrix D as follow.

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

Answer is.

- $B = A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Eigenvalues of B : $\lambda_1 = 1$ $\lambda_2 = 1$
- Eigenvectors of B : $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Singular values: $\sigma_1 = 1$, $\sigma_2 = 1$
- matrix D :

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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Question 3:-

(3) If the eigenvectors of B are not orthonormal, orthonormalise them. Make a matrix V using the orthonormal vectors you obtained. The ordering of the columns of V should be the same as the ordering of the eigenvalues, that is $V = [v_1\ v_2]$. In other words, v_1 is the eigenvector corresponding λ_1 and λ_2 is the eigen vector corresponding to λ_2.

Solution for Question 3.

given the matrix $B = A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have already found its eigenvalue and eigenvectors in part (2).

Step 1: Verify Eigenvectors
The eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$ are.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step 2: Check Orthonormality
we need to verify if these eigenvectors are orthonormal

1. Orthogonality

$$v_1 \cdot v_2 = (1)(0) + (0)(1) = 0$$

the vectors are orthogonal.

2. Normalization

$$\|v_1\| = \sqrt{1^2+0^2} = 1$$
$$\|v_2\| = \sqrt{0^2+1^2} = 1$$

both vectors are already normalized.
since the eigenvectors are already orthormal, it no further orthonormalization is needed.

Step:3. Construct Matrix V.
The matrix V is formed by arranging the orthonormal eigenvectors as columns ordered according to the eigenvalues (which are equal here, so the order is arbitrary)

$$V = [v_1 \ v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So Answer for Question 3 is,

The matrix V is:
$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question (4)

find eigenvalues and eigenvectors of matrix $G = AA^T$, and orthonormalise them. Call them v_1, v_2, and v_3 according to the order of $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of eigenvalues G. Construct the matrix $U = [v_1\ v_2\ v_3]$, and check whether it is an orthogonal matrix.

Solution :

given the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, we first compute $G = A.A^T$.

$$G = A.A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

step 1: find Eigenvalues of G.

The matrix G is diagonal, so its eigenvalues are the diagonal entries.

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 0$$

ordering

$$\lambda_1 = 1 \geq \lambda_2 = 1 \geq \lambda_3 = 0.$$

Step 2 find Eigenvectors of G.

we solve $(G - \lambda I)u = 0$ for each eigenvalue.

1. for $\lambda = 1$:

$$G - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The system $(G - I)u = 0$ gives

$$-u_3 = 0 \Rightarrow u_3 = 0.$$

u_1 and u_2 are free. Thus the eigenvectors are,

$$u = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

A basis for the eigenspace is:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

step 2. continue

for $\lambda = 0$:

$$G = 0I = G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system $Gu = 0$ gives

$u = 0, \quad u_2 = 0$

u_3 is free, Thus the eigenvectors is:

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Step 3: Orthonormalize the Eigenvectors
The eigenvectors u_1, u_2, u_3 are already orthogonal (since C is symmetric) and normalised.

1. $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \|u_1\| = 1$

2. $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \|u_2\| = 1$

3. $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \|u_3\| = 1$

no further orthonormalization is needed.

Step 4: Construct Matrix U
Arrange the orthonormal eigenvectors are columns in the order of $\lambda_1 \geq \lambda_2 \geq \lambda_3$:

$$U = [u_1, u_2, v_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5: Check if U is Orthogonal
A matrix U is orthogonal if $U^TU = I$. Compute.

$$U^TU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, U is an orthogonal matrix.

Final Answer: Question :- 4

- Eigenvalues of A : $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0$
- orthonormal eigenvector of A.

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Matrix U:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check for orthogonality:

U is orthogonal since $U^T U = I$.

Question: 5

find three vectors $\{w_1, w_2, w_3\}$ so that

$$w_i = \frac{1}{\sigma_i} Av_i, \quad i = 1, 2$$

$w_3 \perp w_1$, $w_3 \perp w_2$, and $\|w_3\| = 1$:
Compare $\{w_1, w_2, w_3\}$ with $\{u_1, u_2, u_3\}$ and comment similar characteristics of three sets.

Solution 5:

Objectives:
find three vectors $\{w_1, w_2, w_3\}$ Such that:

1. $w_i = \frac{1}{\sigma_i} Av_i$ for $i = 1, 2$,

2. w_3 is orthogonal to w_1 and w_2 and has unit norm $(\|w_3\| = 1)$.

3. Compare $\{w_1, w_2, w_3\}$ with $\{u_1, u_2, u_3\}$ (from part(4)) and comment on their relationship

Step 1 Recall given quantities:-

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,
- $\sigma_1 = 1$, $\sigma_2 = 1$ (Square roots of eigenvalues of A^TA)
- $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (eigenvectors of A^TA)

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$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(orthonormal eigenvectors of AA^T)

axis

Step 2: Compute w_1 and w_2 using the formula $w_i = \frac{1}{\sigma_i} A v_i$

(i) for $i = 1$:

$$w_1 = \frac{1}{1} A v_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.1 + 0.0 \\ 0.1 + 1.0 \\ 0.1 + 0.0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(ii) for $i = 2$:

$$w_2 = \frac{1}{1} A v_2 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.0 + 0.1 \\ 0.0 + 1.1 \\ 0.0 + 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

observation:
w_1 and w_2 are identical to u_1 and u_2, respectively

Step 3: Compute w_3 (orthogonal to w_1 and w_2)

we need w_3 such that.
1. $w_3 \perp w_1$ (i.e, $w_3 \cdot w_1 = 0$)
2. $w_3 \perp w_2$ (i.e, $w_3 \cdot w_2 = 0$)

3. $\|w_3\| = 1$.

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Approach

- w_1 and w_2 span the xy-plane in $\mathbb{R}^3$
- The orthogonal complement is the z-axis so we choose.

$$w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Verification

- $w_3 \cdot w_1 = 0.1 + 0.0 + 1.0 = 0$
- $w_3 \cdot w_2 = 0.0 + 0.1 + 1.0 = 0$
- $\|w_3\| = \sqrt{0^2 + 0^2 + 1^2} = 1$

Observation:

w_3 is identical to u_3.

Step 4: Compare $\{w_1, w_2, w_3\}$ and $\{u_1, u_2, u_3\}$

The two sets are:-

$$\{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- **Key Observations:**

1. Identical Sets:
both sets are the same, consisting of the standard orthonormal basis of $\mathbb{R}^3$

2. Geometric Interpretation:
   - w_1 and w_2 are the images of the right singular vectors (v_1 and v_2) under A, scaled by $\frac{1}{\sigma_i}$,
   - u_1 and u_2 are the left singular vectors (eigenvectors of AA^T)
   - The equality $w_i = u_i$ reflects the fact that A maps the standard basis vectors directly to the standard basis in $\mathbb{R}^3$

3. Orthonormality:
both sets are orthonormal:
   - $w_i \cdot w_j = \delta_{ij}$ (Kronecker delta)
   - $u_i \cdot u_j = \delta_{ij}$

Step 5: Connection to singular Valule Decoposition (SVD)
The SVD of A is given by:

$$A = U \Sigma V^T.$$

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where :

- $U = [u_1, u_2, u_3] = I_3$ (identity matrix)
- $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,
- $V = [v_1, v_2] = I_2$

Implications.

- The equality $w_i = u_i$ arises because A is already in a "diagonal" form with respect to the standard basis.
- Vectors w_3 and u_3 span the null space of A^T, which is orthogonal to the column space of A.

Final Answer for Question 5

$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

comparison : $\{w_1, w_2, w_3\}$ and $\{u_1, u_2, u_3\}$ are identical.

Both sets form the standard orthonormal basis for $\mathbb{R}^3$ and represents the left singular vectors of A.

Question 6:

Compute the product UDV^T

Solution:

Objective :-

Compute the product UDV^T using the matrixes obtained from previous parts and verify that it reconstructs the original matrix A.

Step 1. : Recall the matrices from previous parts.

from part (3), (2), and (4) we have. :

1. U : Matrix of left singular vectors (from AA^T)

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. D : Diagonal matrix of singular values

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3. V : matrix of right singular vectors (from A^TA)

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, the transpose V^T is

$$V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- Step 2: Compute the product UD
first multiply U and D.

$$UD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The product is computed as follows
  - first column of UD:

(u)

$$\begin{bmatrix} 1.1 + 0.0 + 0.0 \\ 0.1 + 1.0 + 0.0 \\ 0.1 + 0.0 + 1.0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  - Second column of UD:

$$\begin{bmatrix} 1.0 + 0.1 + 0.0 \\ 0.0 + 1.1 + 0.0 \\ 0.0 + 0.1 + 1.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus:

$$UD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Step 3: Compute the final product UDV^T
now, multiply UD by V^T

$$UDV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The product is computed as follows
- first column of UDV^T:

$$\begin{bmatrix} 1\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + 1\cdot 0 \\ 0\cdot 1 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second column of UDV^T

$$\begin{bmatrix} 1\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 1\cdot 1 \\ 0\cdot 0 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus

$$UDV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step : 4 **Verification:**

The original matrix A is.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we observe that : $UDV^T = A$,

final Answer:

$$UDV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = A.$$

Conclusion:

The product UDV^T Successfully reconstruct the original matrix A demonstrating the singular value Decomposition (SVD) of A.

Question 7: Do the questions (1), (2), (3), (4) for the slightly different version of A:

$$A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$$

Construct G_p, U_p, μ_{pi}, V_p, D_p, and U_p. Justify the changes you observe (In this question you may use decimal numbers).

Solution: for matrix $A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$

We will systematically compute the requested quantities using gaussian elimination.

(i) Rank of A_p.
The rank is the number of linearly independent columns (or rows).

- Columns

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

These are linearly independent (neither is a scalar mutiple of the other).

- Rows: The first two are linearly independent, and the third row is multiple of the first row

Conclusion rank $(C(A_p)) = 2$.

2. Basic for the column Space $C(A_p)$

The column space is spanned by the linearly independent columns of A_p.

Basis for $C(A_p) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

3. Basis for the Row Space $C(A_p^T)$
The row space is spanned by the non-zero rows of A_p after Gaussian elimination

Gaussian Elimination:-

$$A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 0.1 R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

the non-zero rows first rows of orriginal matrix

Basis:

Basis for $C(A_p^T) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

4. Basis for the Null Space $N(A_p)$.
The null space consists of all $x \in \mathbb{R}^2$ such that $A_p X = 0$

Solve $A_p X = 0$.
from the reduced row echlon form (RREF) of A_p.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0, \; x_2 = 0.$$

Conclusion
The null space is trivial

Basis for $N(A_p) = \emptyset$ (no basis vectors).

5. Basis for the left Null Space $N(A_p^T)$.
The left Null space consists of all $y \in \mathbb{R}^3$ such that $A_p^T y = 0$.

Solve $A_p^T y = 0$

$$A_p^T = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives
1. $y_1 + 0.1 y_3 = 0 \Rightarrow y_1 \Rightarrow -0.1 y_3$,
2. $y_2 = 0$

The soluton is $\quad y = y_3 \begin{bmatrix} -0.1 \\ 0 \\ 1 \end{bmatrix}, \; y_3 \in \mathbb{R}.$

Basis:

Basis for $N(A_p^T) = \left\{ \begin{bmatrix} -0.1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Verification of Dimensions

1. Column Space + left Null Space.

   1. $\dim(C(A_p)) + \dim(N(A_p^T)) = 2+1 = 3 = m$

2. Row Space + Null space

   2. $\dim(C(A_p^T)) + \dim(N(A_p)) = 2+0 = 2 = n.$

both conditions are satisfied.

~~findings~~ Observation: 1

$$\boxed{\begin{aligned} &\text{rank}(A_p) = 2, \\ &\text{Basic for } C(A_p) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\ &\text{Basic for } C(A_p^T) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\ &\text{Basic for } N(A_p) = \phi, \\ &\text{Basic for } N(A_p^T) = \left\{ \begin{bmatrix} -0.1 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}}$$

Solution for Matrix $A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$

Step 1: Compute $B_p = A_p^T A_p$

first, find. $A_p^T = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix}$

Now multiply A_p^T by A_p:

$$B_p = A_p^T A_p = \begin{bmatrix} 1.1 + 0.0 + 0.1 \cdot 0.1 & 1 \cdot 0 + 0.1 + 0.1 \cdot 0 \\ 0.1 + 1.0 + 0.0.1 & 0.0 + 1.1 + 0.0 \end{bmatrix}$$

Observation 1 $B_p = \begin{bmatrix} 1.01 & 0 \\ 0 & 1 \end{bmatrix}$

Key observation
The perdurbation 0.1 introduces a sligh change to the diagonal entry B_{11}.

Step 2.: Find Eigenvalues (λ) and eigenvectors (v) of B_p

Solve $\det(B_p - \lambda I) = 0$

$$\det \begin{bmatrix} 1.01 - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix} = (1.01 - \lambda)(1 - \lambda) = 0$$

- Eigenvalues
  $\lambda_1 = 1.01$, $\lambda_2 = 1$.

- Eigenvectors:
  - for $\lambda_1 = 1.01$

$$(B_p - 1.01I)v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -0.01 \end{bmatrix} v_1 = 0 \Rightarrow$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - for $\lambda_2 = 1$:

$$(B_p - I)v_2 = 0 \Rightarrow \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

observation:
Eigenvectors remain the standard basis, but λ_1 increasis slighty due to pertubation.

Step 3. Compute singular values (σ_i)

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{1.01} \approx 1.005 \quad \sigma_2 = \sqrt{\lambda_2} = 1.$$

Step 4. Construct Matrix D_p.

$$D_p = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.005 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Verification : The structure matches A_p, with third row zeroed out.

Observation.

$$B_p = A_p^T A_p = \begin{bmatrix} 1.01 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues of B_p : $\lambda_1 = 1.01$, $\lambda_2 = 1$,

Eigenvectors of B_p : $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Singular values : $\sigma_1 \approx 1.005$, $\sigma_2 = 1$

$$D_p = \begin{bmatrix} 1.005 & 0 \\ 0 & 1 \\ 0 & 0. \end{bmatrix}$$

Key observations.

1. Impact of perturbation
   - The eigenvalue λ_1 increases from 1 to 1.01, reflecting the added 0.1 perturbation in A_p
   - Singular value adjust accordingly, with $\sigma_1 \approx 1005$

2. Stability of Eigenvectors:
   The eigenvectors v_1 and v_2 remain unchanged, indicating the perturbation does not alter the principal directions of B_p.

3. Matrix D_p:
   The non-zero singular value σ_1 capture the strech introduced by the perturbation.

This decomposition is critical for understanding how small perturbations affect the singular value decomposition (SVD) of A_p.

now: pertubed matrix $A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$

Step 1. Compute $B_p = A_p^T A_p$.

$$B_p = \begin{bmatrix} 1.01 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2. Eigenvalues and Eigenvectors of B_p.

- Eigenvalues
  $\lambda = 1.01$, $\lambda_2 = 1$.

- Eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Step 3. Orthonormalization.

The eigenvectors v_1 and v_2 are already orthonormal.

- $v_1 . v_2 = 0$

- $\|v_1\| = \|v_2\| = 1$

Step: 4 Construct V_p :

$$V = [v_1 \ v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Observation: 4

Observation for A_p:

Despite the perturbation, the eigenvectors remain orthonormal, and V_p is identical to V. The perturbation only affect the eigenvalues (λ_1 increases to 1.01).

Comparison and Justification of changes

Property	Original A	perturbed A_p	Justification.
eigenvalues(λ)	$\lambda_1 = 1$, $\lambda_2 = 1$	$\lambda_1 = 1.01$ $\lambda_2 = 1$	Perturbation increases λ_1 but leaves λ_2 unchanged.
Eigenvectors (V).	$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	same as A	Perturbation does not alter eigenvector direction.
Matrix V	$V = I$	$V_p = I$	Eigenvectors remain orthonormal in both cases.

key: Insights,

1. Stability of Eigenvectors:
The eigenvectors v_1 and v_2 are inherently orthonormal for both A and A_p, So no additinal orthonormalization is requred.
2. Impact of perturbation.
The pertubation 0.1 in A_p increases a_1 but not affect the eigenvector direction. This shows robustness in the principal directions of B_p.
3. matrix V
V is the Identity matrix in both cases indicating that the row space of A and A_p is perfectly aligned with the standard basis.

Answer

$$V_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$ eigenvalues

$$\lambda_1 = 1.01$$
$$\lambda_2 = 1$$

The perturbation in A_p, increases λ_1 but leaves V_p unchanged, demonstrating that small pertubations do not necessarily disrupt eigen vector orthonormality.

Now: matrix $A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$

Step 1: Compute $G_p = A_p A_p^T$

$$G_p = A_p A_p^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 0.01 \end{bmatrix}$$

Step 2: find Eigenvalues (λ) and eigenvectors (u) of G_p

- Eigenvalues:

Solve $\det(G_p - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda & 0 & 0.1 \\ 0 & 1-\lambda & 0 \\ 0.1 & 0 & 0.01-\lambda \end{bmatrix} = (1-\lambda)[(1-\lambda)(0.01-\lambda) - 0.01] = 0.$$

$$\lambda_1 \approx 1.01, \ \lambda_2 = 1, \ \lambda_3 \approx 0$$

- Eigenvectors:
  - for $\lambda_1 \approx 1.01$ $\quad u_1 \approx \begin{bmatrix} 0.995 \\ 0 \\ 0.1 \end{bmatrix}$
  - for $\lambda_2 \approx 1$ $\quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda_3 \approx 0$; $u_3 \approx \begin{bmatrix} -0.1 \\ 0 \\ 0.995 \end{bmatrix}$

Step 3 Orthonormalize Eigenvectors.

Normalize u_1 and u_3

$$u_1 = \frac{1}{\sqrt{1+\epsilon^2}} \begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix} \approx \begin{bmatrix} 0.995 \\ 0 \\ 0.1 \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{1+\epsilon^2}} \begin{bmatrix} -\epsilon \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.1 \\ 0 \\ 0.995 \end{bmatrix}$$

Step 4 Construct V_p, and verify orthogonality

$$V_p = [u_1 \ u_2 \ u_3] \approx \begin{bmatrix} 0.995 & 0 & -0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 0.995 \end{bmatrix}$$

- check orthogonality

$V_p^T V_p \approx I$, so V_p is orthogonal.

Comparison and Justification of changes.

Observation : 5

Property	Original A	perturbed Ap	Justification
Eigenvalue(λ)	$\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0$	$\lambda_1 \approx 1.01$, $\lambda_2 = 1$, $\lambda_3 \approx 0$	Perturbation increases λ_1 due to added ϵ
Eigenvector(u)	$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$u_1 \approx \begin{bmatrix} 0.995 \\ 0 \\ 0.1 \end{bmatrix}$, $u_3 \approx \begin{bmatrix} -0.1 \\ 0 \\ 0.995 \end{bmatrix}$	Perturbation tilts u_1 and u_3 away from stadard basis.
Matrix U	$U = I$ (Identity)	$U_p \approx \begin{bmatrix} 0.995 & 0 & -0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 0.995 \end{bmatrix}$	U_p becomes a rotation matrix due to the perturbation

Key observations:

1. perturbation effect:
- The eigenvalue λ_1 increases slightly, reflecting the added energy in the perturbed direction.
- The eigenvectors u_1 and u_3 tilt to align with the perturbation, while u_2 remains unchanged.

2. Orthogonality perservation:
both V and V_p are orthogonal, confirming that the SVD Structure is maintained despite the perturbation.

3. Geometric Interpretation:
The perturbation introduces a slight rotation in the column space of A_p, Captured by V_p.

for A_p.

(i) Eigen values of G_p: $\lambda_1 \approx 1.01$, $\lambda_2 = 1$, $\lambda_3 \approx 0$

(ii) Eigen vectors $u_1 \approx \begin{bmatrix} 0.995 \\ 0 \\ 0.1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 \approx \begin{bmatrix} -0.1 \\ 0 \\ 0.995 \end{bmatrix}$

(iii) $V_p \approx \begin{bmatrix} 0.995 & 0 & -0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 0.995 \end{bmatrix}$, orthogonal : Yes.

Questions:-

(8) Construct a low-rank approximation of A and Ap following this definition using their corresponding eigenvectors:

$$A \approx \sigma_1 v_1 u_1^T$$

$$A_p \approx \sigma_{p1} v_{p} u_{p1}^T$$

here σ_{p1}, v_{p1}, and U_{p1} are corresponding to the eigenvalue and eigen vectors related to Ap. Compare these two matrices and justify the changes if you observe any.

Solution:8 :-

low-Rank approximation of A and Ap.

1. Low-Rank Approximation of original matrix A from previous parts we have.

- $\sigma_1 = 1$, $\sigma_2 = 1$
- $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Rank-1 Approximation of A:

$$A \approx \sigma_1 v_1 u_1^T = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Observation:

This approximation captures only the first column of A, losing the second column entirely

2. Low-Rank Approximation of perturbed matrix A_p.

from part (7), we have:

- $\sigma_{p1} \approx 1.005$.
- $v_{p1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $u_{p1} \approx \begin{bmatrix} 0.995 \\ 0 \\ 0.1 \end{bmatrix}$

Rank-1 Approximation of A_p:

$A_p \approx \sigma_{p1} v_{p1} u_{p1}'$

$= 1.005 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.995 & 0, & 0.1 \end{bmatrix} = \begin{bmatrix} 1.005 \cdot 0.995 & 0 & 1.005 \cdot 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Observation

The approximation now includes the perturbation term (0.1) in the third row, the modified structure of A_p

Comparison and Justification of changes
(at) end of key insight,
next page,

Key insight:

The perturbation in A_p introduces a slight tilt in the domainant singular vector (u_{p1}), allowing the rank-1 approximation to partially capture the modified structure. The original A lacks this flexibility, leading to a coarse approximation.

Final Answer:-

Rank-1 approximation of A: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Rank-1 approximation of $A_p \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0.1 & 0 \end{bmatrix}$

Changes.

1. A_p's approximation retains the perturbation (0.1) while A's does not.
2. The singular value σ_{p1} increases slightly reflecting the perturbation's effect,

Sundaram

Comparison and justification of changes,

Property	Original A	Pertubed Ap	Justification.
(1) Singular Value (σ_1)	1	≈ 1.005	Perturbation increases the "strength" of the dominant singular value.
(2) Approximation Quality	Loses second column entirely.	Capture the pertubation in the third row.	The portubation introduces a new direction in Ap preserved in the approximatry
(3) Error.	$\|A - \text{Approx}\|_F = 1$	$\|Ap - \text{Approx}\|_F \approx 1$	Both approximations discard the second singular components but Ap's pertubation is retained.
(4) Visualization,	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0.1 & 0 \end{bmatrix}$	Blue highlights the retained pertubation.

Question:

(9) (For D and HD levels) consider the slightly altered version of A :

$$A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix}$$

(9.1) find eigenvalues and eigenvectors of $A_\epsilon A_\epsilon^T$ and $A_\epsilon^T A_\epsilon$ in terms of ϵ. Compute $\sigma_i = \sqrt{\lambda_i}$ in terms of ϵ for positive eigenvalues

(9.2) Construct V_ϵ, D_ϵ, and U_ϵ

(9.3) Critically analyse the impact of ϵ on the σ_i, λ_i, eigenvectors, V_ϵ, D_ϵ and U_ϵ

Solution for part (9): Analysis of A_ϵ

given matrix:
$$A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix}$$

(9.1) Eigenvalues and eigenvectors of $A_\epsilon A_\epsilon^T$ and $A_\epsilon^T A_\epsilon$

Step 1: Compute $A_\epsilon^T A_\epsilon$

$$\boxed{A_\epsilon^T A_\epsilon = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix} = \begin{bmatrix} 1+\epsilon^2 & 0 \\ 0 & 1 \end{bmatrix}}$$

- Eigenvalues (λ_i):

$$\det(A_\epsilon^T A_\epsilon - \lambda I) = (1+\epsilon^2-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 1+\epsilon^2, \quad \lambda_2 = 1$$

- Eigenvectors (v_i):

- For $\lambda_1 = 1+\epsilon^2$:

$$\begin{bmatrix} 0 & 0 \\ 0 & -\epsilon^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- For $\lambda_2 = 1$

$$\begin{bmatrix} \epsilon^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Singular Value (σ_i):

$$\sigma_1 = \sqrt{1+\epsilon^2}, \quad \sigma_2 = 1$$

Step 2: Compute $A_\epsilon A_\epsilon^T$

$$A_\epsilon A_\epsilon^T = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix} = \begin{bmatrix} 1+\epsilon^2 & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & \epsilon^2 \end{bmatrix}$$

Eigenvalues (λi):

$$\det(A_\epsilon A_\epsilon^T - \lambda I) = -\lambda(1-\lambda)(\epsilon^2-\lambda) + \epsilon^2(1-\lambda) = 0$$

- $\lambda_1 = 1+\epsilon^2, \lambda_2 = 1 ; \lambda_3 = 0$

Eigenvectors (ui):

- For $\lambda_1 = 1+\epsilon^2$

$$u_1 = \frac{1}{\sqrt{1+\epsilon^2}} \begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix}$$

- For $\lambda_2 = 1$:

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- For $\lambda_3 = 0$

$$u_3 = \frac{1}{\sqrt{1+\epsilon^2}} \begin{bmatrix} -\epsilon \\ 0 \\ 1 \end{bmatrix}$$

(9.2) Construct V_ϵ, D_ϵ and U_ϵ

Matrix V_ϵ:

$$V_\epsilon = [v_1 \; v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix D_ϵ:

$$D_\epsilon = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{1+\epsilon^2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Matrix U_ϵ:

$$U_\epsilon = [u_1 \; u_2 \; u_3] = \begin{bmatrix} \frac{1}{\sqrt{1+\epsilon^2}} & 0 & \frac{\epsilon}{\sqrt{1+\epsilon^2}} \\ 0 & 1 & 0 \\ \frac{\epsilon}{\sqrt{1+\epsilon^2}} & 0 & \frac{1}{\sqrt{1+\epsilon^2}} \end{bmatrix}$$

(9.3) Impact of ϵ on Components

Component	impact of ϵ	Critical Analysis
① Singular values (σ_i)	$\sigma_1 = \sqrt{1+\epsilon^2}$ increase with $\sigma_i \sigma_2 = 1$ unchanged.	Measure the "stretch" along principle axes. perturbation amplifies the first singular value.
② Eigen values (λ_i)	$\lambda_1 = 1+\epsilon^2$ grows quadratically; others unchanged.	Reflects the energy in the perturbed direction
③ Eigenvectors (u_i)	u_1 and u_3 tilt by $\arctan(\epsilon)$; u_2 unchanged.	Tilt captures the perturbation's direction For $\epsilon \to 0$ $U_\epsilon \to I$

Components	Impact of ϵ	Critical Analysis.
Matrix V_ϵ	Unchanged (ϵ Independent)	Right singular vectors depend only on row space, unaffected by the perturbation.
Matrix D_ϵ	First diagonal entry scales with ϵ	Directly reflects the amplification of the first singular value.
Matric V_ϵ	Becomes a rotation matrix for the first and third axes, as ϵ increases.	Encodes the perturbation's geometric effect on the column space.

Key observations -

1. For $\epsilon \to 0$;
   - $A_\epsilon \to A_1$ $\sigma_1 \to 1$, $U_\epsilon \to 1$.

2. For $\epsilon >> 0$;
   - $\sigma_1 \approx \epsilon$, U_ϵ approaches a 90° rotation.

3. Stability of V_ϵ:
   - The right singular vectors remain stable, showing robustness in vector row space.

Final Answer:-

(9.1) Eigen values and singular values

$\lambda_1 = 1+\epsilon^2, \lambda_2 = 1, \lambda_3 = 0; \sigma_1 = \sqrt{1+\epsilon^2}, \sigma_2 = 1$

(9.2) Matrices.

$$V_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_\epsilon = \begin{bmatrix} \sqrt{1+\epsilon^2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$U_\epsilon = \begin{bmatrix} \frac{1}{\sqrt{1+\epsilon^2}} & 0 & \frac{-\epsilon}{\sqrt{1+\epsilon^2}} \\ 0 & 1 & 0 \\ \frac{\epsilon}{\sqrt{1+\epsilon^2}} & 0 & \frac{1}{\sqrt{1+\epsilon^2}} \end{bmatrix}$$

(9.3) Impact of ϵ:

σ_1 and λ_1 grow with ϵ, U_ϵ tilts, V_ϵ remains stable.